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TECHNICAL NOTE 2797

A STUDY OF THE TRANSIENT BEHAVIOR OF SHOCK WAVES IN TRANSONIC CHANNEL FLOWS

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SUMMARY

The accuracy of the result obtained in a fundamental paper by Kantrowitz (NACA TN 1225) that a small short-time lowering of the back pressure in steady, shock-free, transonic diffuser flow causes a stationary or trapped shock to form near the critical sonic channel throat is investigated by considering the contribution of a higher-order term in the short-time calculations which was neglected in Kantrowitz's paper. In this more accurate approximation to the short-time effects, the shock is no longer stationary or trapped unless it is supported by a negative steady-flow back pressure. The inclusion of the higher-order term in the short-time calculations avoids the use of approximate quasi-steady-flow considerations for the complete diffuser flow to increase the accuracy of the shock motion, as was required in Kantrowitz's paper. In a broad sense, the present paper offers a firmer basis for the short-time approach originated in Kantrowitz's paper.

The present results transform into those previously reported in NACA TN 1878 for amplitudes that are small compared to the difference between local and critical sonic velocities of the channel flow.

INTRODUCTION

In Kantrowitz's paper (ref. 1), the time-dependent shock behavior produced by lowering the back pressure and the stability of steady shock-free transonic diffuser flows are treated by dividing the time history of the phenomena into short-time effects and long-time effects. The short-time effects are analyzed for a single distinct upstream disturbance or short pulse for which the equations describing the character of unsteady flow are greatly simplified. For additional simplification of the short-pulse equations the highest-order term is neglected. The long-time effects are concerned with transitory phenomena which occur in the interval between the end of short-time phenomena and the final steady flow state in the channel. The calculations indicate that the short-time effect of a small-amplitude expansion pulse, produced by a

small short-time lowering of the back pressure, consists of the formation of a shock from the crest of the expansion pulse which becomes stationary or trapped. In an originally shock-free flow, the occurrence of a stationary or trapped shock unsupported by a negative back pressure is in disagreement with steady-flow solutions for stationary shocks, and the accuracy of shock-velocity considerations has therefore to be increased. The more accurate shock calculations are made in reference 1 by means of an approximate application of the involved long-time effects and the steady-flow back pressure at the end of the diffuser, with the aid of a convenient quasi-steady-flow approach. The result, known from experiments, is that the originally stationary or trapped shock will consume the short-time expansion pulse if the back pressure is that of the shock-free steady diffuser flow and that the shock has to move to a new position if the back pressure is reduced.

In the present paper the increase in accuracy of the shock-velocity calculations is made in a different way. Since the result of the stationary shock in reference 1 is obtained by neglecting a higher-order term in the equations for short-time-pulse motion, the contribution of this term to the order of accuracy of the shock motion should be investigated before the approximated contributions due to the basically different quasi-steady-flow considerations are taken into account. Such an approach shows whether the short-time considerations directly result in a stationary or trapped shock without application of a negative back pressure.

SYMBOLS

a	velocity of sound
u	flow velocity
x	distance along channel axis
A	cross-sectional area of channel
A_{pulse}	pulse area
t	time
P, Q	parameters of characteristic families; used as quantities for measuring amplitudes of unsteady-flow disturbances. Also used as mere labels for distinction between downstream and upstream disturbances. The expressions "disturbance" and "pulse" are used interchangeably in the present paper.

a^* critical sonic channel flow velocity

M Mach number, u/a

$\overline{M} = \frac{u}{a^*}$

$m = \overline{M}^2 - 1$

x_s location of initial shock formation

$\frac{1}{b_1} = \left(\frac{du_o}{dx}\right)_{x=0}$

$\frac{1}{b_2} = \left(\frac{d^2u_o}{dx^2}\right)_{x=0}$

x_{LE} coordinate of leading edge of pulse

x_{TE} coordinate of trailing edge of pulse

u_1 velocity immediately ahead of trailing shock

u_2 velocity immediately behind trailing shock

γ ratio of specific heats

ρ density

p pressure

Subscript:

o steady-flow values

A prime designates deviations from steady channel flow.

ANALYSIS

Background for Present Analysis

As a time-dependent, upstream-moving disturbance or pulse moves through the steady flow gradient in a diffuser that is free of other

disturbances, the shape of the pulse is changed and reflected waves are produced inside the pulse. The growth in amplitude of the reflected downstream waves from zero magnitude at the leading edge of the upstream pulse to the value when leaving the pulse depends on the magnitude of the flow gradient and the time the reflected waves are permitted to grow inside the pulse or on the length of the pulse (see also refs. 1 and 2). For a given steady flow gradient a certain pulse length can always be found in which the values of the reflected amplitudes grown from zero inside the pulse are negligible compared to the original pulse amplitude. The original pulse and its reflected pulse that is built up over this length and leaves the original pulse can thus be treated separately. This separability of original and reflected pulses holds for pulses which are locally superposed on the steady flow gradient (the extent of the local neighborhood depends on the magnitude of the steady flow gradient).

Disturbances of this nature have the great advantage of presenting a case for which the unwieldy equations for unsteady flow disturbances can be solved with relative simplicity. The application of the localized pulses is mainly in the study of disturbances that influence the steady channel flow for only a comparatively short time. In contrast, the long-time effects which lead to the final steady flow state depend on the influence of the complete diffuser flow on the reflections and repeated reflections from the original pulse. Pulses which can be treated independently of their reflections are designated as short in the subsequent development.

The basic equations for motion of isentropic unsteady flow disturbances are conveniently expressed in terms of the characteristic parameters $P = u + \frac{2}{\gamma - 1} a$ and $Q = u - \frac{2}{\gamma - 1} a$ (see refs. 1 and 2) as

$$\frac{dP}{dt} + ua \frac{d \log A}{dx} = 0 \tag{1}$$

$$\frac{dQ}{dt} - ua \frac{d \log A}{dx} = 0 \tag{2}$$

where

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + (u + a)\frac{\partial P}{\partial x}$$

and

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + (u - a)\frac{\partial Q}{\partial x}$$

The velocities of propagation of P and Q are given, respectively, by

$$\frac{dx}{dt} = u + a \tag{3}$$

and

$$\frac{dx}{dt} = u - a \tag{4}$$

Since the unsteady pulses considered represent deviations from steady flow, the quantities u and a are expressed in the form

$$u = u_o + u'$$

$$a = a_o + a'$$

The characteristic parameters $P' = u' + \frac{2}{\gamma - 1} a'$ and $Q' = u' - \frac{2}{\gamma - 1} a'$, based on deviations from steady flow, which are associated with downstream and upstream motions, respectively, in the underlying steady channel flow are conveniently considered as the amplitudes of downstream and upstream pulses. Equation (4) for the motion of upstream Q' pulses indicates that the foot of an upstream pulse will travel with the velocity $u_o - a_o$, whereas a given amplitude Q' travels with the velocity $u - a = (u_o - a_o) + (u' - a')$.

The resulting equations for the motion of upstream pulses with the amplitude Q' are

$$\frac{dQ'}{dt} = \left\{-(1 + M_o)\left(\frac{3 - \gamma}{4} P' + \frac{\gamma + 1}{4} Q'\right) + \left(M_o^2 - 1\right)\left[\frac{\gamma - 1}{8u_o}\left(P'^2 - Q'^2\right) + \frac{P' + Q'}{2M_o} + \frac{\gamma - 1}{4}(P' - Q')\right]\right\} \frac{du_o}{dx} \tag{5}$$

$$\frac{dx}{dt} = u - a = u_o - a_o + \frac{3 - \gamma}{4} P' + \frac{\gamma + 1}{4} Q' \tag{6}$$

These equations are subsequently simplified by the introduction of the short-pulse concept; that is, the amplitudes P' of the reflected waves inside the pulse are considered to be negligible compared to the amplitude Q' of the pulse under consideration. Setting $P' = u' + 5a' = 0$ is equivalent to letting $u' = -5a'$; therefore, $Q' = u' - 5a' = 2u'$. The amplitude of the short pulse can thus be expressed in terms of the velocity deviation u'. For the sake of mathematical simplicity, equation (5) is further reduced for use near $M_o = 1$ by assuming that the amplitude Q' or $2u'$ is small compared with the critical sonic velocity. (This assumption, however, is not a requirement for short pulses.) Equations (5) and (6) are thus reduced to

$$\frac{du'}{dt} = \left[-(1 + M_o)\frac{\gamma + 1}{4} u' + \left(M_o^2 - 1\right)\left(\frac{u'}{2M_o} - \frac{\gamma - 1}{4} u'\right)\right]\frac{du_o}{dx} \tag{7}$$

$$\frac{dx}{dt} = u_o - a_o + \frac{\gamma + 1}{2} u' \tag{8}$$

In spite of the simplification of the equations for pulse motion achieved through the short-pulse concept, the resulting equations are still too complicated to permit simple solutions. It was noted in reference 1 that, in the neighborhood of $M_o = 1$, the higher-order term in equation (7), $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$, may be neglected compared to $(1 + M_o)u'\frac{\gamma + 1}{4}$ since $M_o - 1$ is negligibly small. For the solution of equations (7) and (8), furthermore, a relation between undisturbed quantities u_o, a_o, and x has to be known. In the neighborhood of $M_o = 1$, $u_o - a_o$ is conveniently expressed in terms of the distance x from the critical sonic section of the channel. In reference 1, the Taylor series development of $u_o - a_o$ in x is broken off with terms of the order x. The development of $u_o(x)$ broken off with the term of the order x is

$$u_o(x) = u_o(0) + x\left(\frac{du_o}{dx}\right)_{x=0} = a_o^* + \frac{x}{b_1} \tag{9}$$

where $\frac{1}{b_1} = \left(\frac{du_o}{dx}\right)_{x=0}$ is a negative quantity following the notation of reference 1, where x is positive in the direction of decreasing steady-flow velocity u_o (see also fig. 1).

The quantity a_o is related to u_o by the Bernoulli equation

$$u_o^2 + \frac{2}{\gamma - 1} a_o^2 = \frac{\gamma + 1}{\gamma - 1} a_o{*}^2 \tag{10}$$

Development of the square-root expression for a_o and subtraction from u_o yields

$$u_o - a_o = \frac{\gamma + 1}{2} \frac{x}{b_1} \tag{11}$$

Substitution of equation (11) into equations (7) and (8) without the term $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$ results in

$$\frac{du'}{dt} = -\frac{\gamma + 1}{2} \frac{u'}{b_1} \tag{12}$$

$$\frac{dx}{dt} = \frac{\gamma + 1}{2} \frac{x}{b_1} + \frac{\gamma + 1}{2} u' \tag{13}$$

The solution of this system of equations is

$$\left(x + \frac{b_1}{2} u'\right)u' = \text{Constant} \tag{14}$$

The pulse distortion near $M_o = 1$ thus follows a family of hyperbolas, in which one asymptote has the slope $\frac{du_o}{dx}$ of the velocity gradient and the other is symmetric to it with respect to the x-axis. Figure 1 shows the distortion of an expansion pulse in the velocity plane; an expansion pulse is chosen since its shock formation is of special interest. A pulse with, for example, the original shape $\overline{ABCD}$ is distorted as it approaches $M_o = 1$ and takes the shape ABCD. In the course of this distortion along the family of hyperbolas, the part of the pulse to the

right of the line x_s = Constant will come to lie at a larger distance x from the minimum section of the channel than other parts of the pulse. The overhanging part CEF of the pulse cannot physically exist since, when the crest of the pulse reaches point E, its slope $\frac{du'}{dx}$ is infinite and shock formation therefore occurs. A triangular pulse with a trailing shock is finally formed; the leading expansion phase of the pulse is at the slope $-\frac{du_o}{dx}$.

Because of the occurrence of an infinite slope $\frac{du'}{dx}$, it is desirable to introduce combinations of u' and x which will not undergo such extreme changes as $M_o = 1$ is approached. The pulse area is such a quantity. As is indicated in reference 1, the pulse area is sufficient to determine the position of the shock trailing a triangular pulse near $M_o = 1$, because the shape and movement of the leading expansion phase as it approaches $M_o = 1$ are known and simple. For the case discussed in reference 1 $\left(\text{that is, the term } \left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right) \text{ is neglected}\right)$ the pulse-area growth with time is obtained from equations (12) and (13). The pulse-area growth with time is

$$\frac{d}{dt}\int_{x_{LE}}^{x_{TE}} u'dx = \int_{x_{LE}}^{x_{TE}} \frac{d}{dt}(u'dx)$$

where the differentiation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}$$

may be performed under the integral sign since $\frac{d}{dt}$ indicates a motion with the pulse. From equations (12) and (13),

$$\begin{aligned}\frac{d}{dt}(u'dx) &= \frac{du'}{dt}\,dx + u'\,\frac{d}{dt}\,dx \\ &= -\frac{\gamma + 1}{2}\frac{1}{b_1}u'dx + \frac{\gamma + 1}{2}\frac{1}{b_1}u'dx + \frac{\gamma + 1}{2}u'du'\end{aligned}$$

The first two terms cancel. The integral of the remaining term from the leading edge x_{LE} to the trailing edge x_{TE} of the pulse is zero since the corresponding values of u' are equal (zero) at these points. The pulse-area growth with time when the higher-order term $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$ is neglected is thus zero, the value obtained in reference 1.

Present Analysis

Mathematical development.- In order to analyze the effect of the higher-order term $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$ on the pulse-area growth near the critical sonic channel section, $u_o - a_o$ or $M_o - 1$ is developed in terms of the distance x from the sonic throat. Since $M_o - 1$ is of the order of the distance x made nondimensional and the distance x and the amplitude u' made nondimensional are assumed to be small and of the same order, the lowest-order contribution of the term $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$ is of the order x^2. The condition that u' and x be of equal order is well-satisfied for a triangular pulse bounded by a shock near $M_o = 1$ (u' is proportional to x for such a pulse); the motion of such a triangular shock-bounded pulse is of main interest in this paper. For the present calculations the Taylor series of $u_o(x)$, broken off with terms of the order x^2 rather than x as was done in reference 1, is used:

$$u_o(x) = u_o(0) + x\left(\frac{du_o}{dx}\right)_{x=0} + \frac{x^2}{2}\left(\frac{d^2u_o}{dx^2}\right)_{x=0} \tag{15}$$

In analogy with reference 1, the coefficients are

$$\left(\frac{du_o}{dx}\right)_{x=0} = \frac{1}{b_1}$$

and

$$\left(\frac{d^2u_o}{dx^2}\right)_{x=0} = \frac{1}{b_2}$$

With the aid of the Bernoulli equation (eq. (10)), $u_o - a_o$ is expressed in terms of x as

$$u_o - a_o = \frac{\gamma + 1}{2} \frac{x}{b_1} + \frac{\gamma + 1}{4} \frac{1}{b_2} x^2 + \left[\frac{\gamma - 1}{4} + \frac{(\gamma - 1)^2}{8}\right] \frac{1}{a_o{*}b_1{}^2} x^2 + \ldots \tag{16}$$

In order to obtain $M_o - 1$, equation (16) is divided by a_o. If, for the neighborhood of $M_o = 1$, equation (7) is written in the modified form

$$\frac{du'}{dt} = \left\{-\left[2 + (M_o - 1)\right] \frac{\gamma + 1}{4} u' + 2(M_o - 1)\left(\frac{u'}{2M_o} - \frac{\gamma - 1}{4} u'\right)\right\} \frac{du_o}{dx}$$

the substitution of the expressions for $u_o - a_o$ and $M_o - 1$ into the modified equation (7) and equation (8) yields

$$\frac{du'}{dt} = -\frac{\gamma + 1}{2} \frac{u'}{b_1} - \frac{\gamma + 1}{2} \frac{u'x}{b_2} + \left[-\frac{(\gamma + 1)^2}{8} + \frac{(3 - \gamma)(\gamma + 1)}{4}\right] \frac{u'x}{b_1{}^2 a_o{*}} \tag{17}$$

and

$$\frac{dx}{dt} = \frac{\gamma + 1}{2} \frac{x}{b_1} + \frac{\gamma + 1}{4} \frac{x^2}{b_2} + \left[\frac{\gamma - 1}{4} + \frac{(\gamma - 1)^2}{8}\right] \frac{x^2}{a_o{*}b_1{}^2} + \frac{\gamma + 1}{2} u' \tag{18}$$

The pulse-area growth

$$\frac{d}{dt} \int_{x_{LE}}^{x_{TE}} u'dx = \int_{x_{LE}}^{x_{TE}} \frac{d}{dt}(u'dx)$$

is obtained by using equations (17) and (18). Now

$$\frac{d}{dt}(u'dx) = \frac{du'}{dt} dx + u' \frac{d}{dt} dx$$

$$= -\frac{C_3}{b_1} u'dx - \frac{C_3}{b_2} xu'dx + \frac{C_1}{a_o{*}b_1{}^2} xu'dx +$$

$$\frac{C_3}{b_1} u'dx + \frac{C_3}{b_2} xu'dx + \frac{2C_2}{a_o{*}b_1{}^2} xu'dx + C_3u'du'$$

or

$$\frac{d}{dt}(u'dx) = \frac{C_1 + 2C_2}{a_o{*}b_1{}^2} xu'dx + C_3u'du' \tag{19}$$

where

$$C_1 = -\frac{(\gamma + 1)^2}{8} + \frac{(3 - \gamma)(\gamma + 1)}{4}$$

$$C_2 = \frac{\gamma - 1}{4} + \frac{(\gamma - 1)^2}{8}$$

and

$$C_3 = \frac{\gamma + 1}{2}$$

Since the term $C_3u'du'$ disappears when the integration is performed from the leading edge x_{LE} to the trailing edge x_{TE} of the pulse, the pulse-area growth with time is

$$\frac{d}{dt} \int_{x_{LE}}^{x_{TE}} u'dx = \frac{C_1 + 2C_2}{a_o{*}b_1{}^2} \int_{x_{LE}}^{x_{TE}} xu'dx \tag{20}$$

Now the pulse-area growth with time is calculated near $M_o = 1$, where the pulse has assumed triangular form. The reason the pulse can still be considered in triangular form for the present case in which terms of higher order than those in reference 1 are considered for the pulse-area growth is as follows: In the present approximation the asymptotic velocity distribution of supersonic steady flow approached by the leading edge of the expansion pulse is no longer a straight line symmetric to the linear steady-flow subsonic velocity distribution, because the relation between velocity and cross-sectional area (or distance x) for supersonic and subsonic steady isentropic channel flow is asymmetric with respect to critical sonic velocity (see fig. 2 and the appendix). For the pulse that is cut off by the discontinuous shock at a small value of x, the pulse area added due to the slight deviation of the pulse amplitude u' from symmetric conditions will be of higher order than that being considered (see fig. 2). The pulse-area addition due to the asymmetric condition of the present approximation can thus be neglected and the pulse can be considered in triangular form. The integration in equation (20) over the triangular shape is simplified by the fact that the expansion phase is bounded by the shock which coincides with the ordinate u_1'. The integration therefore only has to be performed over the leading expansion phase from x_{LE} to x_{TE}. In the neighborhood of $M_o = 1$ the leading expansion phase can be considered as parallel to the asymptote $u' = -\frac{2}{b_1}x$ for the present case and is given by $u' = -\frac{2}{b_1}\left(x - x_{LE}\right)$, where b_1 is a negative quantity. Substitution of this expression for u' into equation (20) yields for the triangular-pulse-area growth with time

$$\frac{d}{dt}\int_{x_{LE}}^{x_{TE}} u'dx = -\frac{2}{b_1}\frac{C_1 + 2C_2}{a_o{}^*b_1{}^2}\int_{x_{LE}}^{x_{TE}} x\left(x - x_{LE}\right)dx$$

$$= -\frac{2\left(C_1 + 2C_2\right)}{a_o{}^*b_1{}^3}\left[x_{LE}\frac{\left(x_{TE} - x_{LE}\right)^2}{2} + \frac{\left(x_{TE} - x_{LE}\right)^3}{3}\right] \qquad (21)$$

As $M_o \longrightarrow 1$ or $x_{LE} \longrightarrow 0$ equation (21) becomes

$$\frac{d}{dt}\int_{x_{LE}}^{x_{TE}} u'dx = -\frac{2\left(C_1 + 2C_2\right)}{3a_o{}^*b_1{}^3}x_{TE}{}^3 \qquad (22)$$

Since b_1 is a negative quantity, equation (22) shows that, as the triangular pulse approaches $M_o = 1$, its area grows and is proportional to the cube of the distance from the critical sonic section. Since the area of a triangular pulse A_{pulse} is

$$A_{pulse} = \int_{x_{LE}}^{x_{TE}} u' dx = \frac{x_{TE} - x_{LE}}{2} u_1'$$

$$= -\frac{1}{b_1}\left(x_{TE} - x_{LE}\right)^2 \tag{23}$$

equation (22) can also be stated in the form, as $x_{LE} \longrightarrow 0$,

$$\frac{1}{A_{pulse}} \frac{d\left(A_{pulse}\right)}{dt} = \frac{2}{3} \frac{C_1 + 2C_2}{a_o{*}b_1{}^2} x_{TE} \tag{24}$$

which indicates that the logarithmic growth rate of the pulse area is proportional to the distance of the pulse-area center from the critical sonic section.

The speed of the trailing shock is obtained by comparing the pulse-area growth from equation (21) with the growth obtained by differentiating equation (23) with respect to time

$$\frac{d\left(A_{pulse}\right)}{dt} = -\frac{2}{b_1} \frac{C_1 + 2C_2}{a_o{*}b_1{}^2}\left[x_{LE} \frac{\left(x_{TE} - x_{LE}\right)^2}{2} + \frac{\left(x_{TE} - x_{LE}\right)^3}{3}\right]$$

$$= -\frac{2}{b_1}\left(x_{TE} - x_{LE}\right)\left(\frac{dx_{TE}}{dt} - \frac{dx_{LE}}{dt}\right) \tag{25}$$

where the term $\frac{dx_{LE}}{dt}$ still has to be determined. The speed $\frac{dx_{LE}}{dt}$ of the foot $u' = 0$ of the leading phase of the expansion pulse (see eq. (18)) is

$$\frac{dx_{LE}}{dt} = \frac{\gamma + 1}{2} \frac{1}{b_1} x_{LE} + \text{Constant } x_{LE}{}^2$$

As $M_0 \longrightarrow 1$ or $x_{LE} \longrightarrow 0$, the speed $\frac{dx_{TE}}{dt}$ of the trailing edge of the pulse therefore becomes

$$\frac{dx_{TE}}{dt} = \frac{C_1 + 2C_2}{3a_0{}^*b_1{}^2} x_{TE}{}^2 \tag{26}$$

In terms of the pulse amplitude $u_1' = -\frac{2}{b_1} x_{TE}$, the speed of the trailing edge of the pulse is

$$\frac{dx_{TE}}{dt} = \frac{C_1 + 2C_2}{12} \frac{u_1'^2}{a_0{}^*} = \frac{(\gamma + 1)(3 - \gamma)}{96} \frac{u_1'^2}{a_0{}^*} \tag{27}$$

According to equation (27), the trailing edge moves away from the critical sonic channel section in agreement with the pulse-area growth given by equations (22) and (24).

Correction to increase accuracy of shock velocity.- In the development of equation (27), the motion of the discontinuous trailing edge has been identified with that of the trailing shock, without further discussion of the process in which the shock is obtained from the overhanging pulse. In making no special issue concerning the effect of shock formation on the pulse growth, it is tacitly taken for granted that it does not affect the magnitude of the pulse area as a function of time. In that case the shock is obtained by averaging the pulse area such that the overhanging area CGM is equal to the new area MEN added by the shock (see fig. 1). This averaging process for obtaining shocks is known to apply to disturbances of small amplitudes u' moving through constant cross section (ref. 3). The applicability of this averaging process to the present case of varying cross section is directly connected with the fact that the pulse will remain a shock-bounded triangle as it approaches $M_0 = 1$. More specifically, any time the pulse begins to overhang it is at once averaged by a shock (fig. 1 represents an exaggerated picture of the process of shock formation); that is, the averaging process by the shock occurs in neighboring sections. The shock-wave influence can thus be treated as independent of variation of cross section (the exact proof lies in the fact that the thickness of the shock is zero and its pressures are bounded).

Since an averaging process applies to the small amplitudes u', and equation (27) for the shock speed contains a term of the order $u_1'^2$,

the problem arises whether the averaging process which gives the first-order effect of the shock has to be corrected with a higher-order term. It is indicated in the preceding paragraph that the process of shock formation can be treated independently of the variation in cross section; thus, the higher-order effects in the expressions for shock speed in constant cross section can be used. The development for a downstream shock is (in the notation of ref. 3, eq. (72.05))

$$U = u_o + c_o + \frac{\gamma + 1}{4}(u - u_o) + \frac{(\gamma + 1)^2}{32}\frac{(u - u_o)^2}{c_o} \tag{28}$$

Also, it can be found from reference 3, equations (72.06) and (72.03), that the sum of the terms $u_o + c_o + \frac{\gamma + 1}{4}(u - u_o)$ is equivalent to the average velocity $\frac{1}{2}(u_o + c_o + u + c)$. In other words, the first three terms in equation (28) deal with the averaging of the speed of the overhanging pulse by that of the shock. (Note that, since in the present case upstream waves are being considered, the averaged shock velocity is $\frac{1}{2}(u_o - c_o + u - c)$.)

Introduction of the higher-order effect $\frac{(\gamma + 1)^2}{32}\frac{{u_1'}^2}{a_o^*}$ (in the present notation) effectively increases the strength and speed of the trailing shock which moves upstream during its formation and thus has to be subtracted from the shock speed based on averaging the pulse-area growth (eq. (27)). The proper speed of the trailing shock thus is

$$\frac{dx_{TE}}{dt} = \frac{\gamma + 1}{32}\left[\frac{3 - \gamma}{3} - (\gamma + 1)\right]\frac{{u_1'}^2}{a_o^*} = -\frac{\gamma + 1}{24}\gamma\frac{{u_1'}^2}{a_o^*} \tag{29}$$

The negative speed indicates that the shock tends to consume the short-time expansion pulse as the pulse approaches $M_o = 1$. Note that the ratio of the proper negative shock speed (eq. (29)) and the averaged shock speed (eq. (27)) is $\frac{-4\gamma}{3 - \gamma} = -3.5$ (for air, $\gamma = 1.4$). The proper

negative shock speed is thus several times larger than the averaged positive speed based on the pulse-area growth.

Conditions for stationary trailing shock.- An independent check of sign and magnitude of the present results is simple to make with the aid of steady-flow considerations for a stationary trailing shock. In order to bring the pulse-consuming shock to rest, an upstream-moving pulse area has to be fed into the shrinking pulse area bounded by the moving shock. The amplitude u_2' of this fed-in pulse area (see fig. 2) has to be equal to the back velocity directly behind the stationary shock obtained from steady-flow considerations in the region extending from the critical sonic throat to the rear of the shock (see the appendix). The stationary shock used for such a check does not represent the actual steady-flow solution that has to be adjusted to a certain back pressure at the end of the complete diffuser; however, it represents a possible steady-flow solution for a different back pressure at the end of the complete diffuser. The magnitude of the error in the actual steady-flow back pressure (velocity) at the end of the diffuser depends on the importance of the influence of the long-time effects, governed by the repeated reflections, leading to the steady flow state in the complete diffuser. The present check of the short-time results gains in physical significance because, as discussed in a subsequent section, the long-time effects will exert relatively minor influence.

The back velocity necessary to bring the shock motion to rest is found from the equivalent condition that the sum of the pulse-area shrinkage per unit time due to the shock and the pulse-area gain per unit time due to application of the (still unknown) positive back velocity immediately behind the shock is zero. The amplitude of the reflected downstream pulse is shown subsequently to be negligible compared to the back velocity. The pulse-area shrinkage per unit time of the triangular pulse is given for $x_{LE} \longrightarrow 0$ (eq. (25)) by

$$\frac{d\left(A_{pulse}\right)}{dt} = -\frac{2}{b_1} x_{TE} \frac{dx_{TE}}{dt}$$

Introduction of equation (29) with $u_1' = -\frac{2}{b_1} x_{TE}$ yields for the pulse-area shrinkage $\left(b_1\right.$ is a negative quantity$\left.\right)$

$$\frac{d\left(A_{pulse}\right)}{dt} = \frac{1}{3} \frac{-(\gamma + 1)\gamma}{b_1{}^3 a_o{}^*} x_{TE}{}^3 \tag{30}$$

The rate at which the pulse area due to back velocity enters the triangular pulse at the shock location x_{TE} is given by

$$-\int_0^{u_2'} \frac{dx}{dt}\, du'$$

because the speed at which the various parts of the back-velocity pulse enter the shock is their velocity in the negative x-direction. Equation (18) for $\frac{dx}{dt}$ is then applied, and

$$-\int_0^{u_2'} \frac{dx}{dt}\, du' = -\int_0^{u_2'} \left[\frac{\gamma + 1}{2}\left(\frac{x_{TE}}{b_1} + u'\right) + \text{Constant } x_{TE}^2\right] du'$$

$$= -\frac{\gamma + 1}{2} \frac{x_{TE}}{b_1} u_2' - \frac{\gamma + 1}{4} u_2'^2 - \text{Constant } x_{TE}^2 u_2'$$

If the sum of the triangular-pulse-area shrinkage (eq. (30)) and the pulse-area addition based on the back velocity u_2' is set equal to zero, where $x_{TE} = -\frac{b_1}{2} u_1'$,

$$-u_1'^2\left[\frac{(\gamma + 1)\gamma}{24a_o^*} u_1' + \text{Constant } u_2'\right] + \frac{\gamma + 1}{4} u_2'\left(u_1' - u_2'\right) = 0 \qquad (31)$$

In equation (31) the back velocity u_2' is to be considered negligibly small compared with u_1'. The condition of smallness of u_2' was actually introduced previously by the fact that the shock was assumed to cut off the full height of the pulse triangle. (In order to include the possibility of values of u_2' which are of the order of u_1', the whole process of shock formation would have to be considered, during which the shock would cut off only a finite part of the pulse. For the present problem the final stages of pulse shrinkage, represented by the pulse-area triangle, are, however, of primary interest.) Neglect of u_2' compared with u_1' in equation (31) yields for the back velocity u_2'

$$u_2' = \frac{\gamma}{6} \frac{u_1'^2}{a_o^*} \qquad (32)$$

It was previously stated that the amplitude of the positive back velocity u_2' that will bring the shock to rest would have to be equal to the back velocity required immediately behind a stationary steady-flow shock. This back velocity is obtained by subtracting the velocity increment through a steady shock near $M_o = 1$ from the increment between supersonic and subsonic velocities corresponding to the same channel area near $M_o = 1$ for isentropic flow. In the appendix this difference is shown to be equal to the value in equation (32).

This calculation of the back velocity u_2' is especially simple because the amplitude of the reflected downstream pulse, which is built up in the triangular upstream pulse, can be neglected. Under such conditions the back velocity required to bring the shock (trailing the triangular pulse) to rest does not have to be superposed on the reflected amplitude but can be directly superposed on the foot of the trailing-shock amplitude. The proof for the permissibility of the neglect is now given.

The amplitude $\Delta P'$ of the built-up reflected short pulse near Mach number 1 is obtained from the equation for a downstream or P' pulse. A numerical estimate of the reflected amplitude is given in reference 1; however, its order of magnitude is not specified there. The derivation is thus repeated here with different emphasis. The growth dP' of a reflected P' wave is given by (for air, $\gamma = 1.4$)

$$\frac{dP'}{dt} = -\frac{du_o}{dx}\left[(1 - M_o)\frac{3P' + 2Q'}{5} + \left(M_o^2 - 1\right)\left(\frac{P' + Q'}{2M_o} + \frac{P' - Q'}{10}\right)\right]$$

where $\frac{P'^2 - Q'^2}{20u_o}$ is neglected. For consideration of $\frac{dP'}{dt}$ as the growth of the reflection of a short Q' pulse, P' is neglected compared with Q'; thus, near $M_o = 1$,

$$\frac{dP'}{dt} = \frac{2}{5} Q'(1 - M_o)\frac{du_o}{dx}$$

For a downstream pulse, the relation between t and x is given by the downstream speed of the pulse foot (for the present order of accuracy) as

$$\frac{dx}{dt} = u_o + a_o$$

Introducing $1 - M_o$ = Constant x (eq. (16)) to obtain the first-order effect for the reflected amplitude and noting that $Q' = 2u'$ yields

$$dP' = \text{Constant } u'x \frac{du_o}{u_o + a_o}$$

Near $M_o = 1$, $u_o + a_o \approx 2a_o{}^*$ and writing

$$du_o = \frac{du_o}{dx} dx = \frac{1}{b_1} dx$$

results in

$$dP' = \text{Constant } u'x \, dx$$

The reflected amplitude $\Delta P'$ is obtained by using the relation $u' = \text{Constant}\left(x - x_{LE}\right)$ from the triangular pulse, integrating from x_{LE} to x_{TE}, and letting $x_{LE} \longrightarrow 0$

$$\Delta P' = \text{Constant } u_1{}'^3$$

The reflected amplitude produced by the entropy change in the shock is also proportional to the third power of the velocity change in the shock (ref. 3). The amplitude of the reflected pulse is thus negligible compared with the back velocity u_2' which is of second power in u_1' $\left(\text{provided the flow gradient } \frac{du_o}{dx} = \frac{1}{b_1} \text{ has a reasonable value}\right)$. In contrast, as previously indicated, the pulse-area growth and the related reflected-pulse area cannot be neglected under considerations of the same order.

It should be pointed out that the equations used in this analysis and in reference 1 for the determination of the downstream reflection leaving a short upstream pulse are based on isentropic considerations. They therefore do not apply directly to the pulse-area shrinkage due to the motion of the proper shock but rather to the pulse-area growth averaged by the shock. The positive speed of the averaging shock (eq. (27)) and the negative speed of the proper shock (eq. (29)) are, however, of the same order of magnitude (their ratio of -3.5 is not sufficiently large to affect the order). The reflected amplitude $\Delta P'$ based on motion of

the proper shock is thus of the same order of magnitude as the amplitude $\Delta P'$ based on pulse-area averaging.

DISCUSSION OF RESULTS

The result has been obtained that the consumption of the expansion pulse by the shock does not have to wait for long-time effects depending on repeated reflections and leading to the final steady flow state in the complete diffuser but that the consumption begins as soon as the shock has formed. The speed of pulse consumption was considered for the case in which the shock is already fully formed (for the nature of the shock formation, see fig. 1), and the pulse was conveniently considered to be in the immediate neighborhood of Mach number 1 where the leading expansion phase of the pulse approaches zero speed. The fact, however, that the proper negative speed of the pulse-consuming shock (eq. (29)) is larger by the appreciable factor of 3.5 than the shock speed in the opposite direction based on averaging the positively growing pulse area indicates that the pulse consumption begins when the shock is still in the last stages of formation and the leading expansion phase of the pulse has a small but finite speed.

The long-time effects exert their influence through the repeated reflections from the complete diffuser flow, which in turn affect the shock velocity directly through a change in back velocity immediately behind the shock. The amplitudes of the first downstream reflections have been shown previously to be small compared to the back velocity near Mach number 1. The repeated reflections, which are the basis of the long-time effects, are small compared to the first downstream reflections and they move upstream at a low speed. The negligible contribution of the long-time effects indicates that the back velocity immediately behind the shock obtained from short-time-pulse considerations is essentially equal to that corresponding to the actual steady-flow back pressure at the end of the diffuser. The present higher-order approximation to the short-pulse considerations for small shocks near Mach number 1 thus indicates that shock motion occurs immediately after shock formation and is essentially the same as that due to the actual back pressure at the end of the diffuser. The approximate considerations of the long-time effects and of the actual steady-flow back pressure which had to be made in reference 1 are thus avoided in the present paper.

In reference 1 a procedure that is somewhat similar to that in the present paper is used in that the channel end conditions are also applied directly behind the shock. However, in view of the fact that in reference 1 the increased accuracy of the shock-velocity calculations is not balanced by an increased accuracy in the short-time-pulse equations (7) and (8), a true measure of the negligible smallness of the long-time

effects of repeated reflections is lost. This situation is the actual reason why the increased accuracy of the shock velocity in reference 1 has to be gained through discussion of the various steady-flow conditions at the end of the diffuser and why the approximate treatment of the long-time effects has to be considered as a "simplifying assumption" and the long-time effects still cannot be truly neglected. It should be emphasized that in the present short-time considerations the speed of the shock which consumes the pulse is of the order x^2; whereas the speed of the pulse before the leading expansion phase approaches zero speed is of the order of x or u' (eq. (18)). The present results thus agree with those obtained from approximate quasi-steady-flow considerations in reference 1 in that the speed of pulse consumption for small shocks will be small compared with the speed of pulse approach. The present paper, however, indicates that a stationary shock does not occur unless a negative back pressure is applied and that the shock begins to consume the pulse immediately after the shock is formed. This paper also avoids the introduction of the approximate quasi-steady considerations (dealing with the complete diffuser flow) for proof of the conditions of pulse consumption and thus gives a firmer basis for the fundamental short-time approach in reference 1. The fact that for the diffuser flow the shock is not trapped after formation but tends to consume the expansion pulse directly after formation has bearing on the problem of existence of shocks in transonic flows about bodies.

The conclusions concerning the nature of shock formation do not affect the important result obtained in reference 1 that short compression pulses moving upstream can be compensated by the area of a stationary but still short (as defined herein) expansion pulse which is part of the undisturbed steady supersonic flow bounded by a shock. (For a given shock amplitude, an increase in throat length increases the area of the compensating expansion pulse.)

The preceding results of pulse-area growth are now compared with those in reference 2. For purposes of mathematical facility, equations (7) and (8) were simplified by the restriction $u' \ll u_o - a_o$. The significance of this simplification is quickly seen by substituting equation (8) into equation (7), which results in a differential equation for u' in terms of M_o, for which separation of variables is possible. The restriction, however, has the following effect on the pulse distortion for which the term $\left(M_o^2 - 1\right)u'\left(\frac{1}{2M_o} - \frac{\gamma - 1}{4}\right)$ can be neglected; namely, equations (12) and (13) for the pulse distortion are reduced to

$$\left.\begin{aligned} \frac{du'}{dt} &= -\frac{\gamma + 1}{2}\frac{u'}{b_1} \\ \frac{dx}{dt} &= \frac{\gamma + 1}{2}\frac{x}{b_1} \end{aligned}\right\} \tag{33a}$$

Their solution is

$$xu' = \text{Constant} \tag{33b}$$

For such a distortion the pulse will not overhang (see preceding analysis) as it approaches $M_o = 1$, and no shock formation will occur. Note that the restriction $u' \ll x$ causes equations (33) for the pulse distortion to break down as $x \longrightarrow 0$, because $u' \longrightarrow \infty$; the restriction requires that the distance of the pulse from the sonic channel throat remain of the order x. Athough such a simplified approach will give the proper order of magnitude of the pulse-area growth near $M_o = 1$ without the effects of the shock, it cannot treat the final shrinkage of the pulse bounded by a shock. In order to derive an expression analogous to that appearing in reference 2 for the rate of logarithmic growth of the pulse area, equation (20) is divided by the integrated pulse area. As the leading and trailing edges of the pulse are permitted to approach each other, the resulting quotient of integrals is the location of the center of the pulse (or x) times a constant

$$\frac{d \log \int_{x_{LE}}^{x_{TE}} u'dx}{dt} = \frac{1}{\int_{x_{LE}}^{x_{TE}} u'dx} \frac{d}{dt} \int_{x_{LE}}^{x_{TE}} u'dx$$

$$= \frac{C_1 + 2C_2}{a_o{}^*b_1{}^2} \frac{\int_{x_{LE}}^{x_{TE}} u'x\, dx}{\int_{x_{LE}}^{x_{TE}} u'dx}$$

$$= \frac{C_1 + 2C_2}{a_o{}^*b_1{}^2} x \tag{34}$$

In reference 2, the pulse-area growth with time near $M_o = 1$ is not given; however, the pulse-area growth with Mach number is determined. Equation (34) is reduced to the result in reference 2 as follows: A relation between t and x is obtained from

$$\frac{dx}{dt} = \frac{\gamma + 1}{2} \frac{1}{b_1} x \tag{35}$$

Substitution of equation (35) into equation (34) gives for the pulse-area growth with distance

$$\frac{d \log \int_{x_{LE}}^{x_{TE}} u' dx}{dx} = \frac{C_1 + 2C_2}{\frac{\gamma + 1}{2} a_o^* b_1} \tag{36}$$

Since

$$dx = \frac{dx}{du_o} du_o = b_1 du_o$$

equation (36) becomes

$$d \log \int_{x_{LE}}^{x_{TE}} u' dx = \frac{C_1 + 2C_2 \, du_o}{\frac{\gamma + 1}{2} a_o^*} \tag{37}$$

Furthermore, near $M_o = 1$,

$$\frac{du_o}{a_o^*} = \frac{2}{\gamma + 1} dMo$$

and

$$C_1 = -\frac{(\gamma + 1)^2}{8} + \frac{(3 - \gamma)(\gamma + 1)}{4}$$

$$C_2 = \frac{\gamma - 1}{4} + \frac{(\gamma - 1)^2}{8}$$

Equation (37) becomes

$$\frac{d \log \int_{x_{LE}}^{x_{TE}} u' dx}{dM_o} = \frac{3 - \gamma}{2(\gamma + 1)} \tag{38}$$

In reference 2, the expression $d \log(u'dx)/M_o$ has the same significance as the expression $d \log\left(\int u'dx\right)/dM_o$ of equation (38). In agreement with equation (74) of reference 2, the pulse-area growth with Mach number near $M_o = 1$ is thus equal to $1/3$ for $\gamma = 1.4$ (air). The pulse-area growth is zero for a hypothetical gas with $\gamma = 3$ (one degree of freedom).

A simple physical interpretation of the pulse-area growth can be given by noting that for small disturbances the pulse area $\int u'dx$, if multiplied by the density ρ_o, has the dimension of a momentum. Since the momentum has the dimension of the product of force and time, the growth rate of the pulse area is proportional to a force. In the present case of channel flow the force is represented by the axial component of the normal force on the channel wall. The normal force is given by the integral of the excess pressure p' over the length of the pulse and is of finite value (note the connection of the finite-normal-force integral with the pulse area). As the pulse approaches $M_o = 1$, the slope of the channel wall becomes zero and thus the axial component of the finite normal force becomes zero. It is indicated that, in order that the pulse-area growth with time be zero, the entire pulse would have to move to the critical sonic channel section. Since, of course, for the present triangular pulse only the leading edge, not the center of the pulse, approaches asymptotically the sonic section, the rate of pulse-area growth (exclusive of the shock motion) remains slightly positive.

The simple physical picture using momentum considerations can be also used for an explanation of the pulse-area growth with distance. The pulse-area growth with distance is of the dimension $\frac{\text{Force} \times \text{Time}}{\text{Distance}}$ or $\frac{\text{Force} \times \text{Time}}{(u - a) \times \text{Time}}$. For a pulse (or pulse part) that reaches $M_o = 1$, the force is zero, and the pulse velocity $u - a$ also becomes zero. According to these considerations, the pulse-area growth with distance near $M_o = 1$ should thus be of the order of unity.

CONCLUSIONS

A study of the effects of a small short-time lowering of the back pressure in steady shock-free transonic diffuser flow by means of a higher approximation than in NACA TN 1225 yields the following conclusions:

1. The more accurate approximation of the present paper to the short-time effects shows that the shock is no longer stationary or trapped in the diffuser unless it is supported by a negative steady-flow back pressure; the result thus is no longer in disagreement with steady-flow solutions for stationary shocks.

2. The present short-time calculations avoid the use of approximate quasi-steady-flow considerations for the complete diffuser flow to increase the accuracy of the shock motion, as was required in Kantrowitz's paper (NACA TN 1225). The fundamental considerations in Kantrowitz's paper are thus put on a firmer basis.

3. For short-time pulses with amplitudes that are restricted to values that are small even compared with the difference between local and critical sonic velocities of the channel flow, the present results transform into those previously reported in NACA TN 1878.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 10, 1952.

APPENDIX

THE BACK VELOCITY DUE TO STEADY-FLOW SHOCK LOSSES NEAR $M = 1$

The back velocity is obtained by subtracting the velocity increment through a steady shock near $M = 1$ from the difference between supersonic and subsonic velocities corresponding to the same channel area near $M = 1$ for isentropic flow.

The velocity increment through the shock is expressed in terms of (ref. 3)

$$u_2 = \frac{a_o{*}^2}{u_1}$$

(for notation used, see fig. 2) as

$$\frac{\Delta u_{shock}}{a_o{*}} = \frac{u_1 - u_2}{a_o{*}} = \frac{\bar{M}_1{}^2 - 1}{\bar{M}_1} \tag{A1}$$

where the subscript 1 refers to the supersonic side of the shock and the subscript 2 to the subsonic side. If

$$m_1 = \bar{M}_1{}^2 - 1 \tag{A2}$$

is substituted into equation (A1), the result is

$$\frac{\Delta u_{shock}}{a_o{*}} = \frac{m_1}{\sqrt{1 + m_1}} = m_1 - \frac{1}{2} m_1{}^2 + \ldots \tag{A3}$$

The isentropic channel-area equation (where $A*$ is the critical sonic channel area)

$$\left(\frac{A}{A*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

is altered by making the substitution

$$M^2 = \frac{2\overline{M}^2}{(\gamma + 1) - (\gamma - 1)\overline{M}^2}$$

and by using equation (A2) for $\overline{M}^2$. The isentropic channel-area equation then becomes

$$\left(\frac{A}{A^*}\right)^2 = (1 + m)^{-1}\left(1 - \frac{\gamma - 1}{2} m\right)^{-\frac{2}{\gamma-1}} \tag{A4}$$

Development of equation (A4) results in

$$\left(\frac{A}{A^*}\right)^2 = 1 + \frac{\gamma + 1}{4}\left(m^2 + \frac{\gamma - 3}{3} m^3\right) \tag{A5}$$

Since equal areas are being considered,

$$m_1^2 + \frac{\gamma - 3}{3} m_1^3 = m_2^2 + \frac{\gamma - 3}{3} m_2^3$$

or

$$m_1^2 - m_2^2 + \frac{\gamma - 3}{3}\left(m_1^3 - m_2^3\right) = 0 \tag{A6}$$

Introducing the difference $\delta = m_1 + m_2$, where m_2, being on the subsonic side, is a negative quantity, and dividing by $m_1 - m_2$ in equation (A6) yields

$$m_1 + m_2 + \frac{\gamma - 3}{3}\left(m_1^2 + m_1 m_2 + m_2^2\right) = 0$$

or

$$\delta = -\frac{\gamma - 3}{3} \frac{m_1^2}{1 + \frac{\gamma - 3}{3} m_2} \tag{A7}$$

where the small term $\frac{\gamma - 3}{3} m_2$ can be neglected compared with unity. Now, the isentropic velocity difference is given by

$$\frac{\Delta u_{is}}{a_o^*} = \frac{u_1 - u_2}{a_o^*} = \bar{M}_1 - \bar{M}_2 \tag{A8}$$

The difference $\Delta u_{is}/a_o^*$ is expressed in terms of $\delta = m_1 + m_2$ with $\bar{M}_1$ taken from equation (A2) and with $\bar{M}_2 = \sqrt{1 + m_2}$ as

$$\begin{aligned}\frac{\Delta u_{is}}{a_o^*} &= \bar{M}_1 - \bar{M}_2 \\ &= \sqrt{1 + m_1} - \sqrt{1 + (\delta - m_1)} \\ &= \left(1 + \frac{1}{2} m_1 - \frac{1}{8} m_1^2\right) - \left[1 + \frac{1}{2}(\delta - m_1) - \frac{1}{8}(\delta - m_1)^2\right]\end{aligned} \tag{A9}$$

If terms of order higher than m_1^2 (note that δ is of the order m_1^2) are neglected, equation (A9) becomes, with the aid of equation (A7),

$$\begin{aligned}\frac{\Delta u_{is}}{a_o^*} &= m_1 - \frac{\delta}{2} \\ &= m_1 + \frac{1}{2} \frac{\gamma - 3}{3} m_1^2 \\ &= m_1 - \frac{1}{2} m_1^2 + \frac{\gamma}{6} m_1^2\end{aligned} \tag{A10}$$

The difference $\Delta u_{is} - \Delta u_{shock}$ obtained from equations (A3) and (A10) is

$$\frac{\Delta u_{is} - \Delta u_{shock}}{a_o^*} = \frac{\gamma}{6} m_1^2 \tag{A11}$$

By using equation (A2), equation (A11) can be written as

$$\frac{\Delta u_{is} - \Delta u_{shock}}{a_o^*} = \frac{\gamma}{6}\left(\overline{M}_1^2 - 1\right)^2$$

or, near $\overline{M} = 1$,

$$\frac{\Delta u_{is} - \Delta u_{shock}}{a_o^*} = \frac{\gamma}{6}\left[2\left(\overline{M}_1 - 1\right)\right]^2 = \frac{\gamma}{6}\left[\frac{2\left(u_1 - a_o^*\right)}{a_o^*}\right]^2$$

Since in the present considerations only terms of the order $\left(u_1 - u_2\right)^2 = u_1'^2$ are to be retained, it is correct to assume that for the present purpose $u_1 - a_o^* = -\left(u_2 - a_o^*\right)$ or $2\left(u_1 - a_o^*\right) = u_1 - u_2$ (see fig. 2). The back velocity thus is

$$\frac{\Delta u_{is} - \Delta u_{shock}}{a_o^*} = \frac{\gamma}{6}\frac{\left(u_1 - u_2\right)^2}{a_o^{*2}}$$

or, in agreement with equation (32),

$$u_2' = \frac{\gamma}{6}\frac{u_1'^2}{a_o^*}$$

REFERENCES

1. Kantrowitz, Arthur: The Formation and Stability of Normal Shock Waves in Channel Flows. NACA TN 1225, 1947.

2. Hess, Robert V.: Study of Unsteady Flow Disturbances of Large and Small Amplitudes Moving Through Supersonic or Subsonic Steady Flows. NACA TN 1878, 1949.

3. Courant, R., and Friedrichs, K. O.: Supersonic Flow and Shock Waves. Pure & Appl. Math., vol. I, Interscience Publishers, Inc. (New York), 1948.

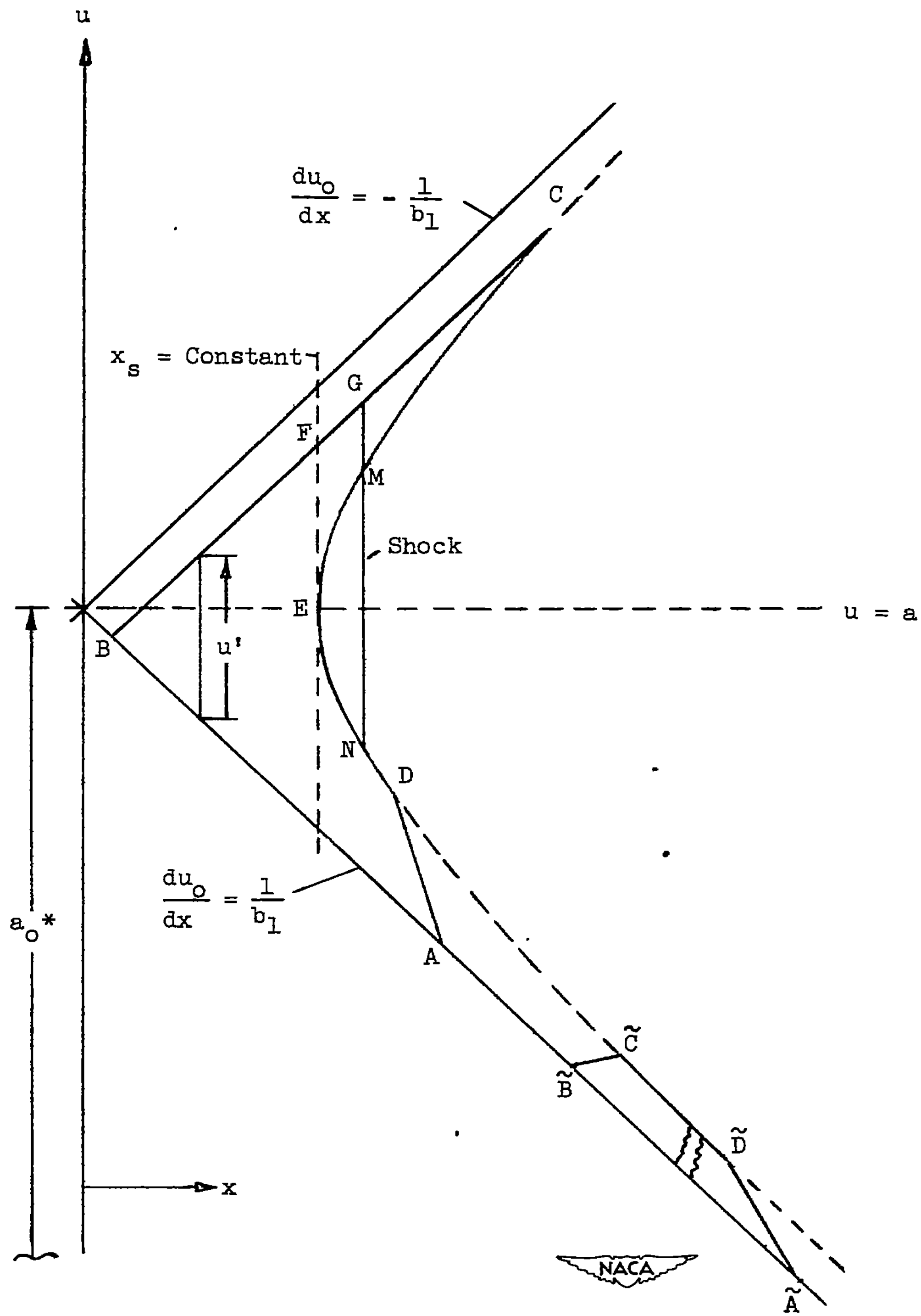

Figure 1.- Distortion of expansion pulse near $M_o = 1$ in the velocity plane.

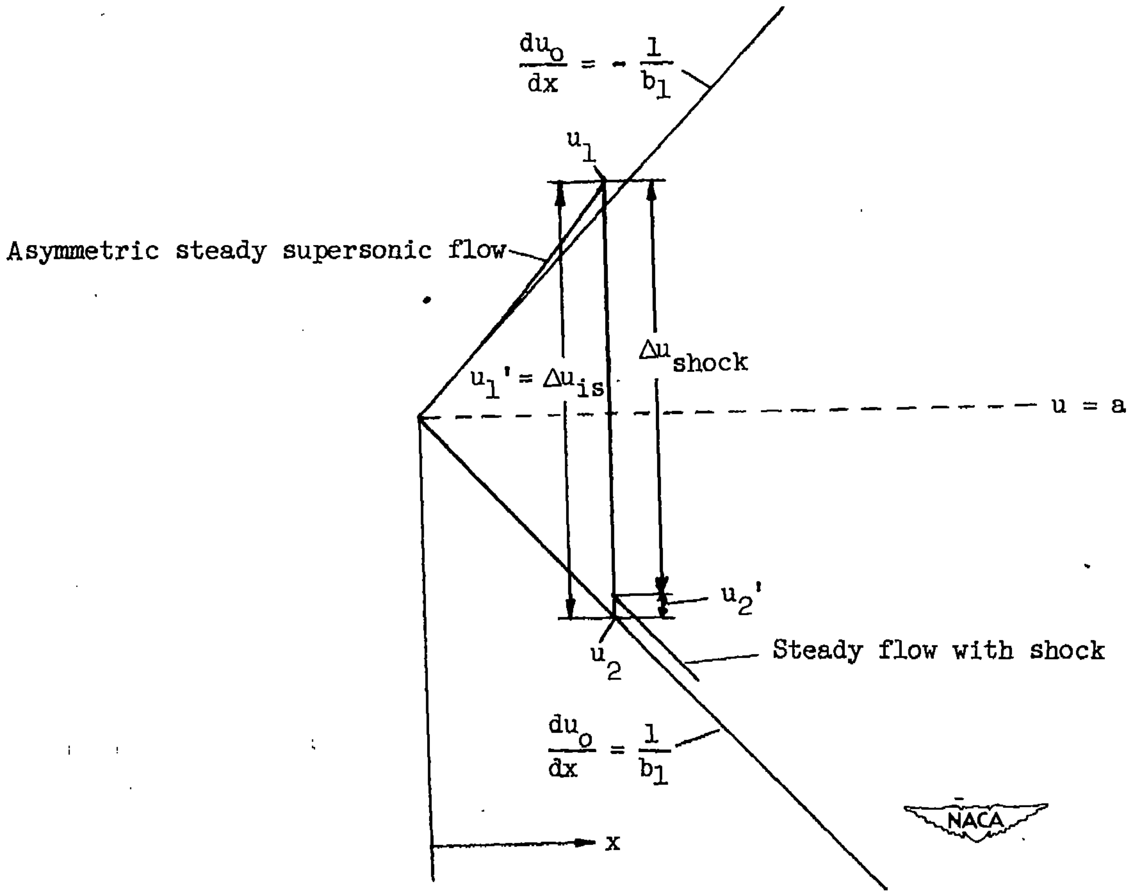

Figure 2.- Conditions for stationary trailing shock.

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